**Modeling Heteroskedasticity with ARCH/GARCH**

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**ABSTRACT**

This project focuses on the implementation of ARCH/GARCH models to effectively model heteroskedasticity in financial returns data. The primary objective is to forecast volatility, providing a crucial measure for assessing and managing risk in financial markets. The analysis elucidates several stylized facts inherent in financial returns data, emphasizing the significance of capturing market uncertainty through conditional heteroskedasticity models. The findings from this research provide valuable insights into enhancing asset management strategies, helping to mitigate the risk of widespread financial shocks and disruptions within the market, and plays a crucial role in clarifying and elucidating the primary tools for modeling heteroskedasticity.

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3. **Introduction**

Regression is often used to model a variable of interest that is dependent on various predictor variables and is stated as follows:

where i = 1,2,3,4 …. n

The objective of this model is to explain Y by minimizing the squared errors and to estimate the respective coefficients. Standard regression theory assumes that the error ε has a zero mean and a constant standard deviation independent of all predictor variables. The variance of these returns is measured by the expectation of the squared error(assuming the expectation is zero):

**.**

As soon as the model is applied to financial time series data where the assumption of independent and constant error terms is relaxed, the estimated error of the coefficients can no longer be used for hypothesis testing and model selection. Hence, we are interested in modeling this error term and forecasting the size of errors which can be considered as a measure of the riskiness of financial assets whose return at time t is Y and the dependent variables are the lagged returns.

1. **Heteroskedasticity**

The riskiness of financial assets can be defined as the variance of returns measured by the size of this error term, which is often referred to as volatility. The size of the error terms is not constant, error terms may reasonably be expected to be larger for some periods than for others and are said to suffer from “heteroskedasticity” (illustrated in Appendix Figure 1). In the context of regression analysis, heteroskedasticity occurs when the variance of the errors (or residuals) is not constant across all levels of the independent variables. This violates one of the assumptions of ordinary least squares (OLS) regression, which assumes that the variance of the errors is constant (homoskedasticity).

Heteroskedasticity can lead to inefficient estimates of the regression coefficients. Standard errors and hypothesis tests based on these standard errors can become unreliable. The presence of heteroskedasticity in regression analysis affects the validity of statistical inferences drawn from the model. Below are two phenomena in financial data that further emphasize the violation of this assumption.

**Volatility Clustering**

Along with the observation that the volatility changes over time, the degree of these changes also shows a tendency to persist forming clusters or groups of high/low magnitude. As the name suggests, volatility clustering refers to the observation that "large changes tend to be followed by large changes, of either sign and small changes tend to be followed by small changes." [[1]](#footnote-1)

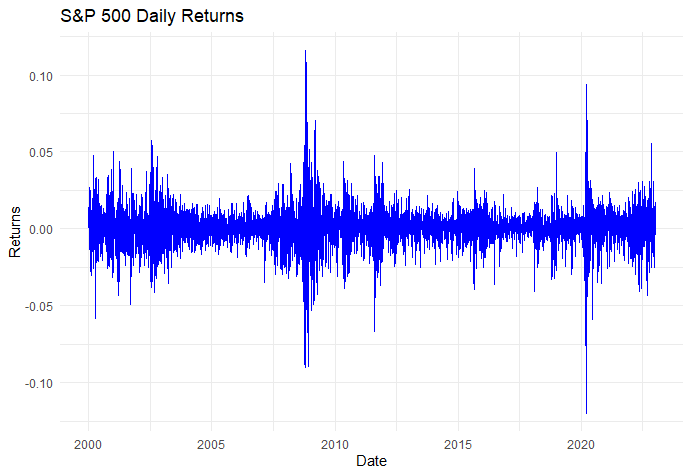


Figure 1-Volatility clustering in SP500 returns

Volatility clustering is often caused by the following reasons:

* **Market Uncertainty**: Increased uncertainty in the market often leads to higher volatility. Uncertainty can arise from economic events, geopolitical tensions, or unexpected news.
* **Information Arrival**: The arrival of new information can trigger volatility clustering. For example, earnings reports, and economic indicators can introduce new information that impacts market expectations.
* **Liquidity Shocks**: Sudden changes in market liquidity, such as during financial crises, can lead to heightened volatility. When liquidity dries up, even small trades can have a significant impact on prices.

**Leverage-effect**

Volatility is a response to the news that the market has not anticipated. However, the timing of the news may not be a surprise, can be anticipated, and gives rise to predictable components of volatility such as economic announcements. Volatility tends to rise or fall in a predictable pattern over time. Black (1976) first noted that changes in stock returns display a tendency to be negatively correlated with changes in returns volatility, i.e., volatility tends to rise in response to “bad news” and to fall in response to “good news”. This phenomenon is termed the “leverage effect”. (Braun 1995) We can observe the phenomenon of “leverage effect” by plotting the market prices and their volatility as below:

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Figure 2-Leverage effect in SP500 returns

Due to the above-explained characteristics of volatility, modeling the volatility of financial assets, such as the S&P 500 index, becomes an important task in quantitative finance. Modeling and forecasting accuracy are generally enhanced with the addition of more data points. However, unlike a clinical trial where you can sample another 1,000 subjects, with time series you simply must wait for each interval (be it a tick, day, month, etc.) to pass. Hence, a prediction model, no matter how rudimentary or advanced, is necessary to digest the historical information and forecast the next day’s value. Given T days of data, the model will digest a certain amount of information, leading to a prediction value for the T+1 day. [[2]](#footnote-2)

1. **ARCH/GARCH Models**

The Autoregressive Conditional Heteroskedasticity (ARCH) / and General Autoregressive Conditional Heteroskedasticity (GARCH) models, are designed to exactly deal with this set of issues. Instead of considering this as a problem to be corrected, ARCH and GARCH models treat heteroskedasticity as a variance to be modeled. As a result, not only are the deficiencies of least squares corrected, but a prediction is computed for the variance of each error term. Modeling time-varying volatility and correlation in financial time series is an important element in pricing equity, risk management, and portfolio management. Higher volatilities increase the risk of assets and higher correlations between assets cause an increased risk in portfolios. ARCH/GARCH models have been applied to model volatility with great success in capturing the heteroskedasticity of financial time series and behaviors such as the leverage effect and volatility clustering.

The ARCH model was first introduced in the seminal paper of Engle (1982). Bollerslev (1986) generalized the ARCH model (i.e. GARCH) by modeling the conditional variance to depend on its lagged values as well as squared lagged values of residuals.

**ARCH model**

In the context of financial returns data, the predictor variables(Xi) are the lagged returns(), and the variable Y is the return at a future time T. The regression, often stated as conditioning, is done to predict future values of the returns based on the present and past values of returns. For example, if we assume that time is discrete, we can express conditioning of returns at time t, as an autoregressive (AR(n)) model:

.

The error term at time t is conditional on the information that, in this example, is represented by the past n values of the returns.

The mean of returns can be modeled by ARMA, and the variance can be modeled by ARCH as follows:

where:

,

with the formulation of the ARCH model being:

Here the coefficients and must be estimated from empirical data. Thus, ARCH(p) is a forecasting model since it forecasts the error variance at time *t* based on the known information in the form of squared residuals of previous p times.

**GARCH Model:**

A useful generalization of this model is the GARCH parameterization introduced by Bollerslev (1986). This model is also a weighted average of past squared residuals. The most widely used GARCH specification asserts that the best predictor of the variance in the next period is a weighted average of:

1) the long-run average variance ,

2) the variance predicted for this period () ,

3) the new information in this period that is captured by the most recent squared residual ().

Such an updating rule is a simple description of adaptive or learning behavior from past information. Therefore, a GARCH (p,q) model for variance looks like this:

The analyst must estimate the coefficients . Then, updating simply requires knowing the previous forecasted or actual q values of and p values of the residual .

The GARCH(p,q) model described above derives its name from the fact that the symbols p and q in parentheses are a standard notation in which the first refers to the number of autoregressive lags (or ARCH terms) that appear in the equation and the second refers to the number of moving average lags specified (often called the number of GARCH terms).

The process is quite straightforward: For any set of parameters and a starting estimate for the variance of the first observation, which is often taken to be the observed variance of the residuals, it is easy to calculate the variance forecast for the second observation. The GARCH updating formula takes the weighted average of the unconditional variance, the squared residual for the first observation, and the starting variance and estimates the variance of the second observation. This is input into the forecast of the third variance, and so forth. Eventually, an entire time series of variance forecasts is constructed (Bollerslev 1986).

**ARCH vs. GARCH**

Below is a simulation of ARCH(1) with and GARCH(1,1) with to illustrate the effect of including the GARCH term in volatility modeling. It can be observed in Appendix Figure 2 that the variance becomes more sensitive and accurate as we include the past information of variance on the model. At times of high volatility, the difference can be seen to be more significant and hence the GARCH model adds complexity to the model for capturing volatility.

To confirm that the estimation methods for these models are valid, 20000 data points were generated from the models and were fitted 150 times iteratively by sampling out 15000 time-series data points to verify if the estimated coefficients are close to the simulation parameters. Below is a table for comparison (R output in Appendix-Results 1) of the average of estimated coefficients and the true values of used in model simulation. The distribution of estimates GARCH is shown below in figure.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **ARCH Model** | **ARCH Estimates** | **GARCH Model** | **GARCH Estimates** |
| **Omega** | 0.1 | 0.0993 | 0.1 | 0.101 |
| **Alpha** | 0.7 | 0.718 | 0.7 | 0.703 |
| **Beta** |  |  | 0.3 | 0.295 |

Table 1 - parameter and estimates comparison

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**GARCH Model on Financial Data**

To model volatility in financial data, the SP500 index is extracted for 10 years from *2012-01-01* to *2022-01-01*. Daily returns are calculated using the closing prices and the ADF (Augmented Dickey-Fuller) test is performed on the daily returns which gives a p-value less than 0.01 (Appendix, result 2) and thus, we can conclude that the returns are stationary, and we can implement an ARMA(0,0) model assuming constant mean. Further, the Breusch-Pegan test for heteroskedasticity is performed and a p-value close to 0 was obtained (Appendix, result 3), concluding the presence of heteroskedasticity in the returns over time. The GARCH model is an extension of ARCH model as it incorporates the weighted average of past variances also. To implement the GARCH model, the following variations are considered:

1. **Distribution Assumption**

The GARCH model makes distribution assumptions about the residuals and the mean return. However, financial time series data often do not follow a normal distribution as extreme positive and negative values are observed at times of shock. To be more representative of real financial data we can specify the model’s distribution assumption to be a Skewed student's t-distribution (SSTD).

Originating from the original Student-t distribution, Hansen (1994) proposed the skewed-t distribution. By adding the skewness parameter, the skewed-t distribution can better describe the asymmetric and fat tail features of financial asset data as compared to the normal distribution. It is defined by two parameters: degrees of freedom (shape) and skewness.

1. **Asymmetricity**

Glosten-Jagannathandan-Runkle (1993) introduced the asymmetric GARCH model, the GJR-GARCH model. The advantage of the GJR-GARCH model is that it can measure volatility due to the different effects of bad news and good news. The GJR-GARCH is defined as follows:

where ,

This is the same as a GARCH(p,q) model, the only difference is that a term has been added to account for negative returns. If the returns are less than a threshold , then the squared residuals are given more weight by an additional parameter and the coefficient associated with negative shock becomes . This is how the GJR-GARCH model captures asymmetric volatility clustering.

Based on the above, a GJR-GARCH model was fitted using the skewed student-t distribution and with constant mean assumption. The estimated parameters are shown inthe Appendix (result 4) and are used to represent the following volatility equation for time T+1, using the information at time T:

Using the above equation, the volatility is forecasted using a fixed rolling window forecast method and is plotted with respect to returns as shown below. We can see that the model is highly accurate in forecasting the volatility as the red line moves in vicinity to the returns indicating higher volatility at riskier times. The model accuracy will be tested using VaR and QPS later.

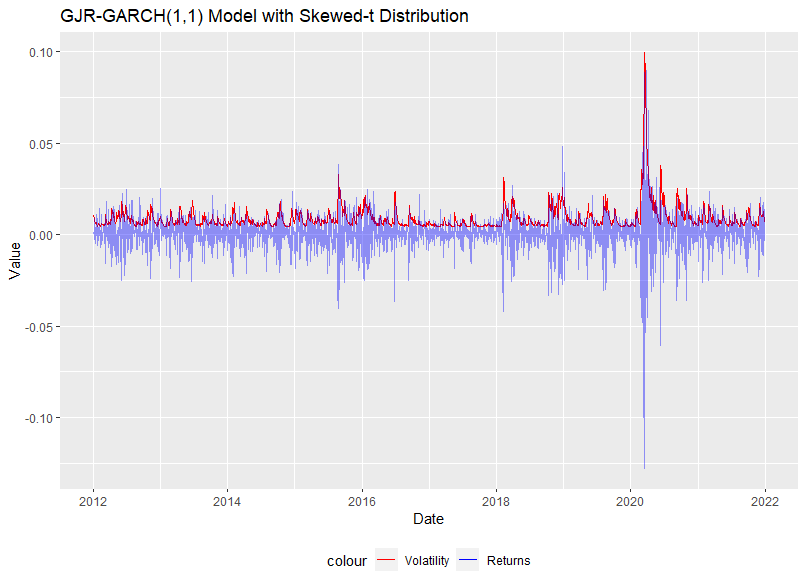
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Figure 3 - Forecasted volatility vs. returns

1. **Value at Risk**

ARCH/GARCH models play a key role in Value at Risk calculations. VaR is a statistical measure used to quantify the level of financial risk within a firm's investments or portfolio. Using ARCH/GARCH models, financial institutions can estimate the potential losses (or gains) at different confidence levels, providing valuable insights into the risks faced by their portfolios.

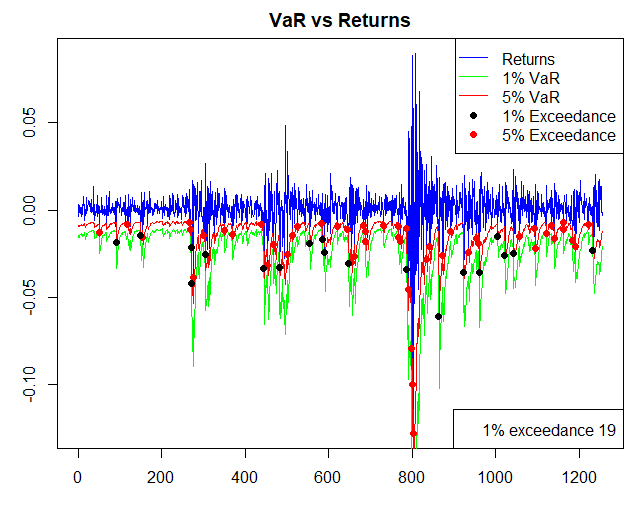
Three ingredients:

1. Portfolio value
2. Time horizon
3. Probability

VaR in risk management terms:For example,1-day 1% VaR of $1 million means that there is 1% probability the portfolio will fall in value by 1 million dollars or more over a 1-day period. Stated the other way, there is 99% probability that the portfolio will not lose the calculated value or more in one day period.[[3]](#footnote-3) Exceedance points are the time points at which the returns are more negative than the calculated value of VaR, which means that the loss was more than the forecasted value at risk: in this case the model is underestimating the risk.

**Dynamic VaR with GARCH**

Var is calculated using the forecasted volatility dynamically for 3 days at a time by fixed rolling window method: new data points are added while old ones are dropped from the sample. Here, historical data from 2012-2017 is used for the first estimation and the GJR-GARCH model is fitted on a window of 3 years data and volatility is forecasted for next 3 days and so on. The process is illustrated in Appendix, Figure 4. Below is a plot of returns vs. forecasted VaR and the exceedance points are marked. Back testing the model is performed by comparing the calculated VaR with the actual realized volatility of historical data and results are reported in Appendix (result 5). For 5 years of data, the total expected exceedance points are 13 but the forecasted VaR exceeds 19 times, implying that the observed exceedance is 1.5% compared to the nominal 1% level. Thus, the model appears to be underestimating the risk, discounting for statistical variability.



1. **Back testing results:**

Quadratic Performance Score (Sukono 2019) is used to measure the performance of VaR that has been estimated. If 𝑟𝑡 is the return at time 𝑡 and 𝑉a𝑅𝑡 is a VaR prediction at time 𝑡, the following size-adjusted frequency approach was introduced by Lopez in 1998:

The QPS statistic, which measures the VaR risk performance by the average deviation of exceedance points is calculated using the following formula:

Where n is the number of datapoints, 𝑝 is a probability value of VaR(0.01). The QPS value is in the [0,2] range with 0 being the minimum value that occurs when 𝑟𝑡 ≤ 𝑉a𝑅𝑡 for all values of t, and 2 is the maximum value that occurs when 𝑟𝑡 > 𝑉a𝑅𝑡 for all values of t. VaR performance is said to be good when the QPS approaches 0. The QPS obtained for this dataset was 0.06609967 which is sufficiently close to 0.

1. **Results and Conclusion**

In this project, heteroskedasticity is addressed in the context of financial data by applying the GJR-GARCH model with a constant mean to estimate Value at Risk (VaR), using a decade-long dataset of S&P 500 index returns spanning from 2012 to 2022. The primary objective was to enhance risk measurement and forecasting performance through the consideration of the leverage effect – a phenomenon capturing the asymmetrical impact of positive and negative shocks on volatility.

The empirical findings reveal a robust performance in modeling volatility, as evidenced by the relatively small Quantile Probability Score (QPS). This metric attests to the efficacy of VaR as a tool for assessing and managing financial risk associated with stock market investments. The QPS results suggest that the GJR-GARCH model effectively captures the intricate dynamics of the S&P 500 index returns, providing a reliable basis for risk evaluation.

**References:**

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7. Braun, Phillip A., Daniel B. Nelson, and Alain M. Sunier. “Good News, Bad News, Volatility, and Betas.” *The Journal of Finance* 50, no. 5 (December 1995): 1575–1603. <https://doi.org/10.1111/j.1540-6261.1995.tb05189.x>.

**Appendix**

**Result 1:**

**A close-up of a computer code

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**Result 2:**

A screenshot of a computer

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**Result 3:**

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**Result 4:**

**A screenshot of a computer

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**Result 5:**

A screen shot of a report

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**Figure 1:**

A diagram of a graph

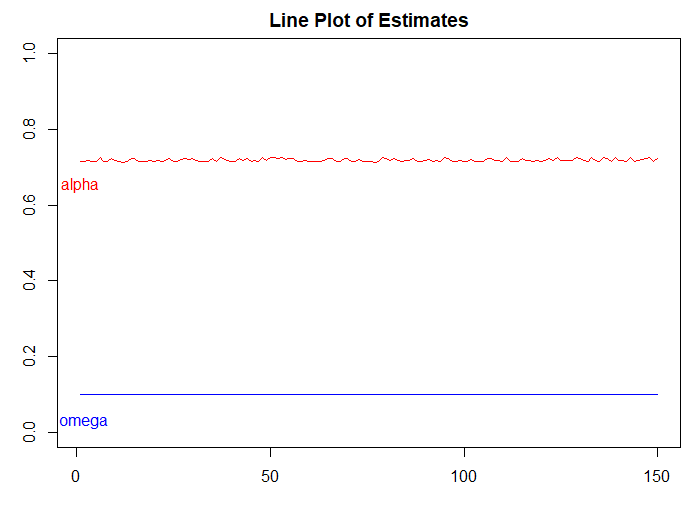
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**Figure 2:**

A graph of a graph

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**Figure 3:**



**Figure 4:**

**A graph of progress bar

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1. <https://en.wikipedia.org/wiki/Volatility_clustering#:~:text=In%20finance%2C%20volatility%20clustering%20refers,be%20followed%20by%20small%20changes.%22> [↑](#footnote-ref-1)
2. ^ https://saltfinancial.com/insights/blog/stack-the-deck-with-rolling-realized-volatility/ [↑](#footnote-ref-2)
3. <https://goldinlocks.github.io/ARCH_GARCH-Volatility-Forecasting/#GARCH-rolling-window-forecast> [↑](#footnote-ref-3)