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**1) OLS and origin of heteroskedasticity**

OLS stands for Ordinary Least Squares, which is a method for estimating the parameters in a Multiple Linear Regression (MLR) model. A multivariate linear regression represents a relationship of proportionality between dependent(y) and multiple independent variables(xi) and is modeled as follows:

where i = 1,2,3,4 …. n,

The above linear regression model states that the expectation of the variable y is β times E(X) plus a constant α. The proportionality relationship between y and x is not exact but also contains an error ε. Standard regression theory assumes that the error ε has a zero mean and a constant standard deviation σ independent of all variables. The standard deviation is the square root of the variance, which is the expectation of the squared error:

It is a positive number that measures the size of the error.When the expectedsize of this error is constant and does not depend on the sizeof the variable x, we call this assumption, homoskedasticity.

**Heteroskedasticity**

Financial return data, in which the variances of the error terms are not equal, error terms may reasonably be expected to be larger for some points or ranges of the data than for others, are said to suffer from heteroskedasticity, which literally means “various dispersion” in ancient Greek. In the context of regression analysis, heteroskedasticity occurs when the variance of the errors (or residuals) is not constant across all levels of the independent variables. This violates one of the assumptions of ordinary least squares (OLS) regression, which assumes that the variance of the errors is constant (homoskedasticity).

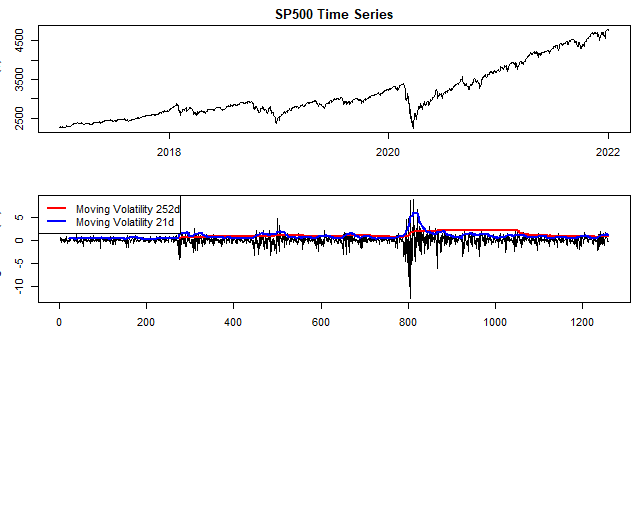
Heteroskedasticity can lead to inefficient estimates of the regression coefficients. Standard errors and hypothesis tests based on these standard errors can become unreliable. If heteroskedasticity is present in your regression analysis, it can affect the validity of statistical inferences drawn from the model.

**Volatility**

Volatility is a statistical measure of asset return dispersion across time used in finance. The standard deviation or variance of price returns is frequently used to calculate it. We’ll use the term “volatility” to refer to both standard deviation and variance in this course. Volatility is used to characterize the uncertainty surrounding the price movement of financial assets. It’s a crucial notion that is used in risk management, portfolio optimization, and other areas. It’s also one of the most active fields of empirical finance and time series analysis study. The risk and volatility of assets are directly proportional to each other which means the riskier a financial asset is, the higher the volatility.

**Leverage-effect**

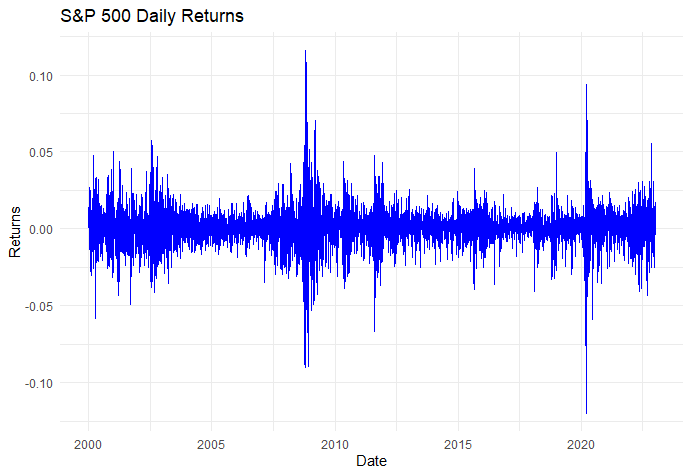
Volatility is a response to the news which must be a surprise. However, the timing of the news may not be a surprise and gives rise to predictable components of volatility such as economic announcements. Volatility tends to rise or fall in a predictable pattern over time. Black (1976) first noted that changes in stock returns display a tendency to be negatively correlated with changes in returns volatility, i.e., volatility tends to rise in response to “bad news” and to fall in response to “good news”. This phenomenon is termed the “leverage effect”. We can observe the phenomenon of “leverage effect” by plotting the market prices and their volatility as below:



**Volatility Clustering**

In time series modeling, it’s a frequent assumption that volatility remains constant throughout time. Looking at financial data suggests that some time periods are riskier than others; that is, the expected value of the magnitude of error terms at sometimes is greater than at others. Moreover, these risky times are not scattered randomly across quarterly or annual data. Instead, there is a degree of autocorrelation in the riskiness of financial returns. Financial analysts, looking at plots of daily returns such as in Figure 1, notice that the amplitude of the returns varies over time and describe this as “volatility clustering.” The ARCH/GARCH models, which stand for autoregressive conditional heteroskedasticity and generalized autoregressive conditional heteroskedasticity, are designed to deal with just this set of issues. The goal of such models is to provide a volatility measure – like a standard deviation -- that can be used in financial decisions concerning risk analysis, portfolio selection, and derivative pricing.

Modeling and predicting the volatility of financial assets, such as the S&P 500 index, is an important task in quantitative finance and is crucial for risk management, derivatives pricing, and portfolio optimization.



**ARCH/GARCH Models**

There is an increasing need to forecast and analyze the size of the errors of the model. Instead of considering this as a problem to be corrected, ARCH and GARCH models treat heteroskedasticity as a variance to be modeled. As a result, not only are the deficiencies of least squares corrected, but a prediction is computed for the variance of each error term. Modeling time-varying volatility and correlation in financial time series is an important element in pricing equity, risk management, and portfolio management. Higher volatilities increase the risk of assets and higher correlations between assets cause an increased risk in portfolios.

ARCH/GARCH models have been applied to model volatility with great success in capturing stylized facts of financial time series mentioned above, such as leverage effect and volatility clustering. The Autoregressive Conditional Heteroskedasticity (ARCH) model was first introduced in the seminal paper of Engle (1982). Bollerslev (1986) generalized the ARCH model (GARCH) by modeling the conditional variance to depend on its lagged values as well as squared lagged values of disturbance.

ARCH model

In financial and economic models, conditioning is often stated as regressions of the future values of the variables based on the present and past values of the same variable. For example, if we assume that time is discrete, we can express conditioning of returns as an autoregressive (AR(n)) model:

The error term is conditional on the information that, in this example, is represented by the present and the past n values of the returns.

The return can now be modeled as:

Where:

,

And the formulation of the ARCH model is:

where the coefficients must be estimated from empirical data. Thus, ARCH(p) is a forecasting model as it forecasts the error variance at time *t* based on the known information of previous p times.

GARCH Model:

A useful generalization of this model is the GARCH parameterization introduced by Bollerslev (1986). This model is also a weighted average of past squared residuals, but it has declining weights that never go completely to zero. In its most general form, it is not a Markovian model, as all past errors contribute to forecast volatility. The most widely used GARCH specification asserts that the best predictor of the variance in the next period is a weighted average of

1) the long-run average variance,

2) the variance predicted for this period, ()

3) the new information in this period that is captured by the most recent squared residual ().

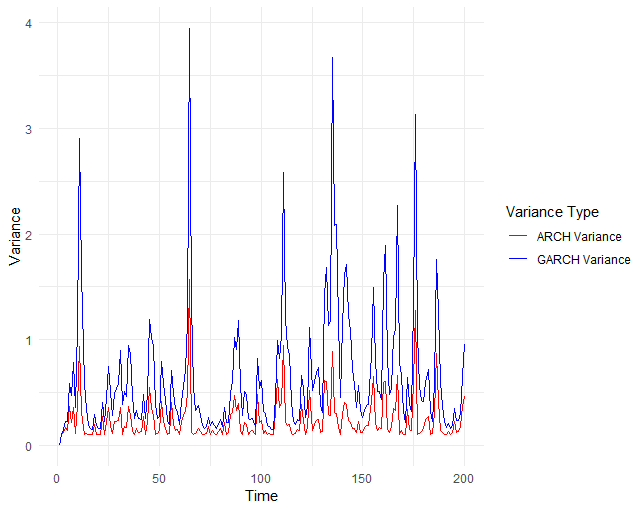
Such an updating rule is a simple description of adaptive or learning behavior from past information. Therefore, a GARCH (1,1) model for variance looks like this:

The econometrician must estimate the coefficients . Then, updating simply requires knowing the previous forecast and the residual .

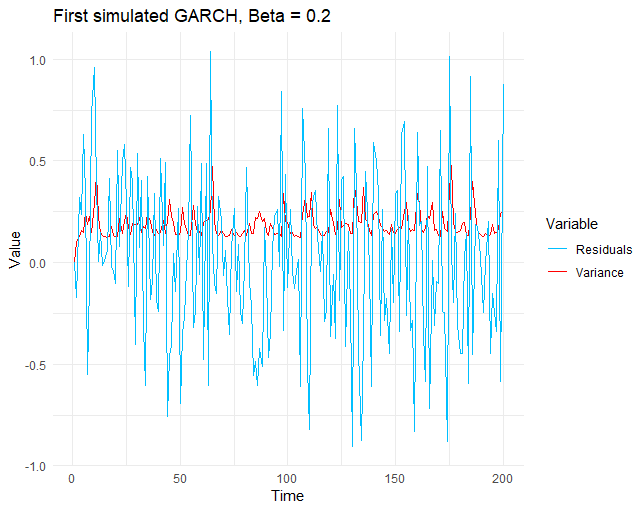
The GARCH model described above and typically referred to as the GARCH (p,q) model derives its name from the fact that the p,q in parentheses is a standard notation in which the first number refers to the number of autoregressive lags (or ARCH terms) that appear in the equation and the second number refers to the number of moving average lags specified (often called the number of GARCH terms). The process is quite straightforward: For any set of parameters and a starting estimate for the variance of the first observation, which is often taken to be the observed variance of the residuals, it is easy to calculate the variance forecast for the second observation. The GARCH updating formula takes the weighted average of the unconditional variance, the squared residual for the first observation, and the starting variance and estimates the variance of the second observation. This is input into the forecast of the third variance, and so forth. Eventually, an entire time series of variance forecasts is constructed.

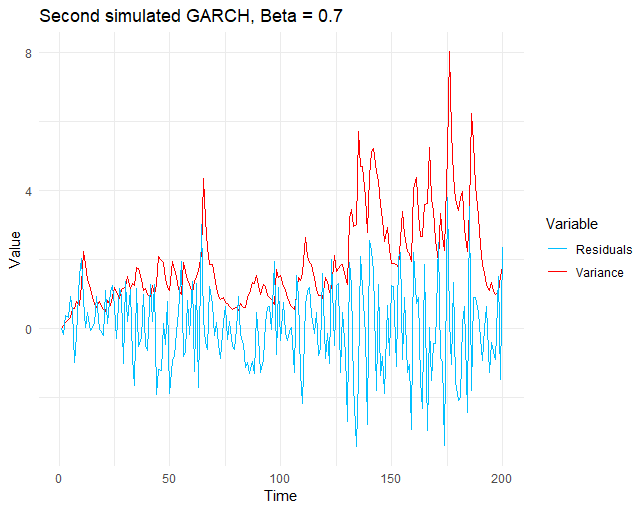
**ARCH vs. GARCH**

Below is a simulation of ARCH(1) and GARCH(1,1) to illustrate the effect of including the GARCH term in volatility modeling. It can be observed in the figure below that the variance becomes more sensitive and accurate as we include the past information of variance on the model. At times of high volatility, the difference can be seen to be significant and hence the GARCH model captures the volatility more accurately.

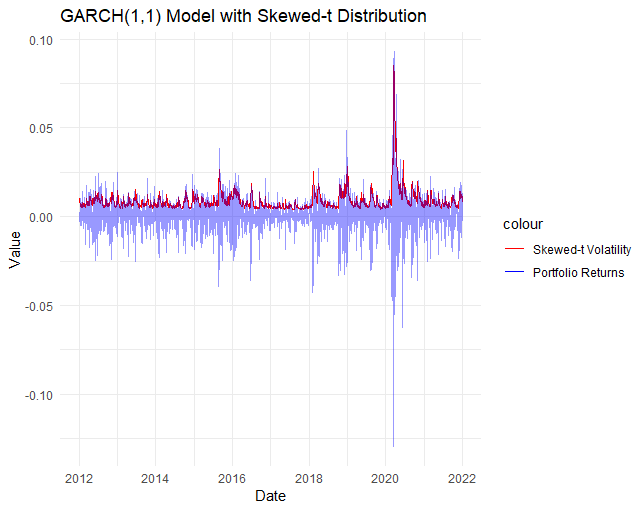
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Impact of model parameter Beta





**GARCH model with skewed t-distribution**

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**Value at Risk**

ARCH/GARCH models play a key role in Value at Risk calculations. VaR is a statistical measure used to quantify the level of financial risk within a firm's investments or portfolio. Using ARCH/GARCH models, financial institutions can estimate the potential losses (or gains) at different confidence levels, providing valuable insights into the risks faced by their portfolios.

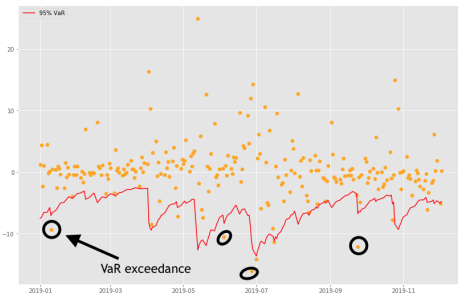
Three ingredients:

1. Portfolio value
2. Time horizon
3. Probability

For example,1-day 5% VaR of $1 million means that there is 5% probability the portfolio will fall in value by 1 million dollars or more over a 1-day period. Stated the other way, there is 95% probability that the portfolio will lose $5000 or less in one day period.

**VaR in risk management terms**

Suppose a 5% daily VaR and 252 trading days in a year. A valued VaR model should have less than 13 VaR exceedance in a year, that is 5% \* 252 if there are more exceedances the model is underestimating the risk.



**Dynamic VaR with GARCH**

* GARCH gives more realistic estimation of VaR
* VaR = mean + (GARCH vol) \* quantile

**Results and Conclusion**

**Limitations**

when using GARCH models for financial analysis, especially in scenarios where asymmetric responses, non-symmetric distributions, and the interpretation of volatility shocks are critical considerations, researchers in the field of finance need to be aware of the following limitations:

if we were able to make conditional forecasts of returns, then the ARCH model describes the behavior of the errors and it is no longer true that the unconditional variance of errors coincides with the unconditional variance of returns. Thus, the statement that ARCH models describe the time evolution of the variance of returns is true only if returns have a constant expectation.

N**eglect of Volatility Asymmetry**: GARCH models assume that only the magnitude of unanticipated excess returns influences volatility, not their positivity or negativity. However, empirical evidence suggests that volatility tends to rise more sharply in response to bad news (negative excess returns) than to good news (positive excess returns). This suggests a limitation of GARCH models in capturing asymmetric responses to positive and negative shocks.

**Symmetric Distribution Assumption**: If the distribution of the error term zt is symmetric, the change in variance is uncorrelated with excess returns. This means that GARCH models assume that volatility changes are not influenced by the direction of excess returns (positive or negative). This assumption might not align with the observed behavior of financial markets.

**Nonnegativity Constraints**: GARCH models include constraints to ensure that the conditional variance remains nonnegative. While this ensures stability in the model, it limits the possibility of capturing certain types of random behavior in the variance process. Additionally, these constraints can complicate the estimation process.

**Difficulty in Estimation**: The nonnegativity constraints can lead to difficulties in estimating GARCH models. Researchers have had to impose specific structures on model coefficients to prevent them from becoming negative, indicating potential challenges in the estimation process.

**Interpretation of Volatility Shocks**: GARCH models are used to analyze the persistence of shocks to conditional variance. The duration for which these shocks persist is crucial in understanding their impact on asset pricing and investment decisions. Long-lasting volatility shocks can significantly influence risk premia and long-term investment strategies.